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The fundamental group of a real flag manifold

by Mark Wiggerman

University of Nijmegen, Department of Mathematics, Toernooiveld, 6525 ED Nijmegen, the Netherlands
e-mail: mwiggerman@aab.e-mail.com

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ABSTRACT

The fundamental group of a real flag manifold is computed. We use the structure of a CW-complex induced by the Bruhat decomposition to determine a presentation of the fundamental group. This generalizes a construction of Ehresmann [4].

1. INTRODUCTION

A real flag manifold is the quotient space G/P of a real, connected, semisimple Lie group G and a subgroup P of a special type which is called a parabolic subgroup. Classical examples are the projective space and the Grassmann manifold.

Topological properties of (real) flag manifolds have been a research topic for a long time. In the 1930's Ehresmann [4] computed the fundamental and homology groups of some classical (i.e. $G = \mathrm{SL}(n, \mathbb{R})$) flag manifolds and recently Kocherlakota [6] computed the homology groups of real flag manifolds in terms of the Dynkin diagram of G. In this paper we will determine a presentation of the fundamental group of a real flag manifold in terms of the Dynkin diagram of the restricted roots and their multiplicities.

Let S denote the set of simple restricted roots of G. The parabolic subgroups are determined (up to conjugation) by a subset F of S and will be denoted by P_F. The main result of this paper is the following theorem.

Theorem 1.1. *Let G be a real semisimple connected Lie group, F a subset of the simple roots and P_F the standard parabolic subgroup of G corresponding to F. Define S^* as the subset of S consisting of simple roots of multiplicity* 1. *The fundamental group of G/P_F has the following presentation: Generators: t_α for every $\alpha \in S^*$; Relations: $t_\beta t_\alpha = t_\alpha (t_\beta)^{\varepsilon(\alpha,\beta)}$ for $\alpha, \beta \in S^*$ and $\alpha \neq \beta$, and $t_\alpha = e$ for each $\alpha \in S^* \cap F$.*

Here $\varepsilon(\alpha,\beta)$ is defined as $(-1)^{(\beta,\alpha^\vee)}$. The numbers $(\beta,\alpha^\vee)$ are the Cartan integers, which determine the Dynkin diagram of the restricted root system.

An important property of a flag manifold is that it has a decomposition called the Bruhat decomposition which gives it the structure of a CW-complex. This structure gives us a presentation (see Corollary 3.2) of the fundamental group provided we can do certain calculations on the 2-skeleton.

The structure of this paper will be as follows: In Section 2 we will introduce the flag manifold and its Bruhat decomposition. In Section 3 we will summarize the necessary theory of CW-complexes and we will analyze the relevant part of the decomposition. In Section 4 we will prove Theorem 1.1 by doing the necessary calculations in a special case which are then generalized for an arbitrary flag manifold. Finally, in Section 5 we will give the fundamental group of the flag manifolds with G split and simple, and $F = \emptyset$.

At this point I would like to thank Erik van den Ban. He suggested this problem to me for my 'doktoraalskriptie'[1] and coached me through it. Also I would like to thank Gert Heckman for his suggestions and discussions when I completed this work.

2. LIE THEORY AND NOTATION

Let G be a connected real semisimple Lie group with Lie algebra $\mathfrak{g}$. Fix a Cartan involution θ of $\mathfrak{g}$ and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ be the corresponding decomposition into the $+1$ and -1 eigenspaces. Fix a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{s}$ and let $\Sigma = \Sigma(\mathfrak{g},\mathfrak{a}) \subset \mathfrak{a}^*$ be the restricted root system with corresponding root space decomposition:

$$\mathfrak{g} = \mathfrak{m} + \mathfrak{a} + \sum_{\alpha \in \Sigma} \mathfrak{g}_\alpha$$

where $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Fix a set of positive roots $\Sigma^+ \subset \Sigma$. This determines the simple roots $S \subset \Sigma^+$. Put $\mathfrak{n} = \sum_{\alpha>0} \mathfrak{g}_\alpha$ and $\bar{\mathfrak{n}} = \sum_{\alpha<0} \mathfrak{g}_\alpha$.

Define K, A, N and $\bar{N}$ to be the analytic subgroups of G with Lie algebra $\mathfrak{k}, \mathfrak{a}, \mathfrak{n}$ and $\bar{\mathfrak{n}}$ respectively. Define M and M^* as the centralizer and the normalizer of $\mathfrak{a}$ in K. We have $G = KAN$, the Iwasawa decomposition. The Weyl group $W = W(\mathfrak{g},\mathfrak{a})$ of Σ is generated by the orthogonal reflections s_α in the hyperplanes $\ker(\alpha)$ and is naturally isomorphic to M^*/M.

[1]The closest, but still incorrect translation is 'master thesis'.

For details on the following definitions of parabolic subalgebra and parabolic subgroup, see [9].

Definition 2.1. A subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is called a *parabolic subalgebra* if its complexification $\mathfrak{p}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$ contains a maximal solvable subalgebra of $\mathfrak{g}_{\mathbb{C}}$.

For a subset F of the simple roots S define $\Sigma(F)$ as $\Sigma \cap (\mathbb{Z} \cdot F)$, the root system in Σ spanned by F. Put

$$\mathfrak{p}_F = \mathfrak{m} + \mathfrak{a} + \sum_{\alpha \in \Sigma^+ \cup \Sigma(F)} \mathfrak{g}_\alpha.$$

Then $\mathfrak{p}_F$ is a parabolic subalgebra of $\mathfrak{g}$. Moreover, up to conjugation these are all parabolic subalgebras of $\mathfrak{g}$. The subalgebra $\mathfrak{p}_F$ is called a standard parabolic subalgebra of $\mathfrak{g}$.

Definition 2.2. A subgroup of G is called a *parabolic subgroup* if it is the normalizer $N_G(\mathfrak{p})$, of a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ in G.

The map $\mathfrak{p} \mapsto N_G(\mathfrak{p})$ from parabolic subalgebras of $\mathfrak{g}$ to parabolic subgroups of G is a bijection. The parabolic subgroup corresponding to $\mathfrak{p}_F$ is denoted as P_F and is called a standard parabolic subgroup.

Definition 2.3. Let G be a real semisimple connected Lie group and $P \subset G$ a parabolic subgroup, then the quotient space G/P is called a *real flag manifold.*

Let $\tilde{G}$ be another connected semisimple Lie group with Lie algebra $\tilde{\mathfrak{g}}$ and a parabolic subgroup $\tilde{P}$ corresponding to $\tilde{\mathfrak{p}} \subset \tilde{\mathfrak{g}}$. Let $\phi : \mathfrak{g} \to \tilde{\mathfrak{g}}$ be an isomorphism mapping $\mathfrak{p}$ onto $\tilde{\mathfrak{p}}$. Then ϕ induces a diffeomorphism $G/P \to \tilde{G}/\tilde{P}$. So any topological property of G/P is determined by the pair $(\mathfrak{g}, \mathfrak{p})$. For this reason it is possible to express the fundamental group in terms of the Dynkin diagram.

For the rest of this paper let P be a minimal parabolic subgroup $P_\emptyset$ of G unless explicitly stated otherwise. We have the decomposition $P = MAN$ which is called the Langlands decomposition. The inclusion $K \hookrightarrow G$ induces a diffeomorphism $K/M \to G/P$. Fix a subset $F \subset S$. Denote by $W(F)$ the subgroup of W generated by s_α with $\alpha \in F$. The inclusion $M^* \hookrightarrow G$ induces a bijection

$$W/W(F) \to P\backslash G/P_F.$$

For $w \in W$ we have $Pw \cdot P_F = Nw \cdot P_F$. The decomposition of G/P into N-orbits:

$$G/P_F = \coprod_{w \in W/W(F)} Nw \cdot P_F \tag{1}$$

is known as the *Bruhat decomposition.* The N-orbits can be parameterized in a nice way.

Lemma 2.4. *The map* $\sum \mathfrak{g}_\alpha \to Nw \cdot P_F, X \mapsto \exp(X)w \cdot P_F$ *is a diffeomorphism. The summation in the domain is taken over all roots* $\alpha > 0$ *with* $w^{-1}\alpha \in \Sigma^- \backslash \Sigma(F)$.

For a proof see DKV [3] Proposition 3.6. So the N-orbit through w is an open cell. They are called *Bruhat cells.* Let us write $C_F(w) = C(w)$ for $Nw \cdot P_F$. Denote the closure of $C(w)$ in G/P_F as $V_F(w) = V(w)$. It can be proved that the $V(w)$ are real algebraic varieties (see [3] Lemma 4.3), they are called *Schubert varieties.*

Let us compute the dimension of the Bruhat cells. For $\alpha \in \Sigma$, let $m_\alpha = \dim(\mathfrak{g}_\alpha)$ be its multiplicity.

Corollary 2.5. $\dim(C(w)) = \sum m_\alpha$, *where the summation on the right hand side is over the same set of roots as in Lemma 2.4.*

There is a system of representatives $W^F \subset W$ of $W/W(F)$, consisting of $w \in W$ such that w is the (unique) element in the coset $wW(F)$ of minimal length.

Corollary 2.6. *Let* $w \in W^F$ *have the reduced expression* $w = s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_q}$. *Then* $\dim(C(w)) = \sum_{i=1}^{q} m_{\alpha_i} + m_{2\alpha_i}$.

The Bruhat order on the Weyl group can be defined as follows (see Deodhar [2]).

Let $v, w \in W$. If for any reduced expression $w = s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_q}$ there is an integer k and a sequence $1 \le i_1 < i_2 < \cdots < i_k \le q$ of integers such that $v = s_{\alpha_{i_1}}s_{\alpha_{i_2}}\cdots s_{\alpha_{i_k}}$ then we say $v \le w$. The Bruhat order has a geometrical interpretation:

Proposition 2.7. $V(w) = \bigcup_{v \le w} C(v)$.

For a proof see Borel [1] or Steinberg [8].

3. CW-COMPLEXES

3.1. Theory

The following has been taken from Massey [7] Chapter 7. A topological space X has the structure of a (finite) CW-complex if there is a sequence

$$X^0 \subset X^1 \subset X^2 \subset \cdots$$

of closed subspaces such that

(i) X^0 is discrete,

(ii) X^n is obtained from X^{n-1} by adjunction of a finite number of n-cells,

(iii) $X = \bigcup_{n=0}^{\infty} X^n$.

Item (ii) means that: $X^n \backslash X^{n-1}$ is a finite disjoint union of n-cells C_λ^n, with λ in some index set L_n. Moreover, for each C_λ^n there is a continuous map $T_\lambda^n : D^n \to \overline{C_\lambda^n}$, called *the characteristic map* of C_λ^n, such that T_λ^n restricted to B^n is a homeomorphism onto C_λ^n and it maps S^{n-1} into X^{n-1}. Here D^n, B^n and S^{n-1} are the subsets of $\mathbb{R}^n$ consisting of x with $\|x\| \leq 1$, $\|x\| < 1$ and $\|x\| = 1$ respectively. The subspace X^n is called the *n-skeleton* of X.

We make the following assumption about the structure of the CW-complex on X:

(CW 1) The 0-skeleton consists of one element: $X^0 = \{x_0\}$.

(CW 2) x_0 lies in the closure of every 2-cell.

These assumptions are not necessary for the main result (Theorem 3.1) but they will make the application to the flag manifold easier.

Because of the assumption (CW 1) we see that every $\overline{C_\lambda^1}$ is homeomorphic to a circle and the characteristic map $T_\lambda^1 : [-1,1] \to \overline{C_\lambda^1}$ is a loop whose homotopy class generates $\pi_1(\overline{C_\lambda^1}, x_0)$. The 1-skeleton is a bouquet of circles intersecting at x_0. Denote by t_λ the homotopy class of T_λ^1 in X^1. Then $\pi(X^1, x_0)$ is a free group with t_λ, $\lambda \in L_1$ as its generators.

Fix a loop $\phi : [-1,1] \to S^1$ whose homotopy class generates $\pi_1(S^1)$. Because of the assumption (CW 2) we have that for every 2-cell C_μ^2, $\mu \in L_2$ we have $x_0 \in \overline{C_\mu^2}$. In particular $x_0 \in T_\mu^2(S^1)$. After composing T_μ^2 with a rotation of D^2 we can arrange that $R_\mu = T_\mu^2 \circ \phi : [-1,1] \to X^1$ is a loop in X^1 starting at x_0. The loop R_μ encircles the 2-cell C_μ^2 exactly once. Let us call this loop a *boundary loop*. Denote by $r_\mu \in \pi_1(X^1, x_0)$ the homotopy class of R_μ in X^1 and call it a *boundary word* of T_μ^2. The boundary words are unique up to conjugation and taking the inverse.

We can now state the result about CW-complexes that will be used in the next section. It must be noted that the assumptions (CW 1) and (CW 2) are not necessary for the validity of the next theorem.

Theorem 3.1. *The inclusion $X^1 \hookrightarrow X$ induces a surjective homomorphism of groups: $\pi_1(X^1, x_0) \to \pi_1(X, x_0)$. Its kernel is the normal subgroup generated by the r_μ, $\mu \in L_2$.*

For a proof see for instance Massey [7] Chapter 7, Theorem 2.1.

Corollary 3.2. *A presentation of $\pi_1(X, x_0)$ is*

$$\langle t_\lambda, \quad \lambda \in L_1 \,|\, r_\mu, \quad \mu \in L_2 \rangle.$$

3.2. Application to the flag manifold

The Bruhat decomposition (1) provides the flag manifold G/P with the structure of a CW-complex, where X^n is defined as the union of the Bruhat cells of dimension less than or equal to n. Moreover, this structure satisfies the assumptions (CW 1) and (CW 2) we made about CW-complexes.

Corollary 2.6 enables us to determine the cells of dimension 0, 1 and 2. First put $S^* = \{\alpha \in S \mid m_\alpha = 1\}$.

Proposition 3.3. *Let* $w \in W$ *then*

(i) $w \in W^F$ *and* $\dim(C(w)) = 0$ *iff* $w = e$.

(ii) $w \in W^F$ *and* $\dim(C(w)) = 1$ *iff* $w = s_\alpha$ *with* $\alpha \in S^*\backslash F$.

(iii) $w \in W^F$ *and* $\dim(C(w)) = 2$ *iff (exactly) one of the following statements holds:*

(a) $w = s_\alpha$ *with* $\alpha \in S\backslash F$, $m_\alpha = 2$, $m_{2\alpha} = 0$.

(b) $w \mp s_\alpha s_\beta$ *with* $\alpha, \beta \in S^*\backslash F$, $\alpha \neq \beta$.

(c) $w = s_\alpha s_\beta$ *with* $\alpha \in S^* \cap F$, $\beta \in S^*\backslash F$, $\alpha \not\perp \beta$.

Before we begin the proof, remark that W^F can be characterized as:

(2) $\qquad w \in W^F \Leftrightarrow w(F) \subset \Sigma^+.$

In turn this implies for subsets F_1 and F_2 of S:

(3) $\qquad F_1 \cap F_2 = \emptyset \Rightarrow W(F_1) \subset W^{F_2}.$

Proof. We will only prove the last assertion. Let $w \in W^F$ with $\dim(C(w)) = 2$. From Corollary 2.6 and part (i) of this proposition, we see that $l(w)$ is either 1 or 2. Consider the case where $l(w) = 1$. Write $w = s_\alpha$. Because $w \in W^F$ we have $\alpha \notin F$. With Corollary 2.6 we see $m_\alpha + m_{2\alpha} = 2$. In general $m_\alpha = 1$ implies $2\alpha \notin \Sigma$ (see Helgason [5] p. 530). Therefore $m_\alpha = m_{2\alpha} = 1$ is not possible and we conclude that $m_\alpha = 2$ and $m_{2\alpha} = 0$. Thus we have shown that statement (iii a) holds.

Now consider the case that $l(w) = 2$. Let $w = s_\alpha s_\beta$ be a reduced expression. Corollary 2.6 states that $m_\alpha + m_{2\alpha} + m_\beta + m_{2\beta} = 2$. From this we see that $m_\alpha = m_\beta = 1$ and $m_{2\alpha} = m_{2\beta} = 0$. Because $w \in W^F$ we have $\beta \notin F$. If $\alpha \notin F$ then statement (iii b) holds. If $\alpha \in F$ then the assumption that $\alpha \perp \beta$ implies that s_α and s_β commute. But in that case clearly $s_\alpha s_\beta \notin W^F$. This contradiction shows that in this case statement (iii c) holds.

Now we prove the converse. It is trivial to show that statement (iii a) implies that $w \in W^F$ and $\dim(C(w)) = 2$. Assume statement (iii b) holds then we have from (3) that $w \in W(S\backslash F) \subset W^F$. Corollary 2.6 now gives us $\dim(C(w)) = m_\alpha + m_\beta = 2$. Finally assume that statement (iii c) holds. In view of Corollary 2.6 it will be sufficient to prove that $w \in W^F$. Using (2) and (3) we can prove that $w(F\backslash\{\alpha\}) \subset \Sigma^+$. To prove that $w(\alpha) \in \Sigma^+$, write $s_\beta(\alpha) = \alpha - n_1\beta$ and $s_\alpha(\beta) = \beta - n_2\alpha$ for n_1, n_2 Cartan integers: We have

(4) $\qquad w(\alpha) = s_\alpha s_\beta(\alpha) = (n_1 n_2 - 1)\alpha - n_1\beta.$

Because α and β are simple non-perpendicular roots, we have $n_i < 0$. Therefore the coefficients of α and β in (4) are positive. This shows that $w \in W^F$. $\square$

Now that we know the 2-skeleton, we proceed to make the computations necessary to obtain boundary words for the 2-cells.

4.1. A special case

Let us first compute the fundamental group of a flag manifold in a special case: Assume that $F = \emptyset$ and $m_\alpha = 1$ for every $\alpha \in S$ (G is split). Later we will see these results can be easily generalized for an arbitrary flag manifold. We will give characteristic maps for the 1- and 2-dimensional cells from which the boundary words can be obtained easily. From Proposition 3.3 we see that the 1-cells are $C(s_\alpha)$ with $\alpha \in S$ and the 2-cells are $C(s_\alpha s_\beta)$ with $\alpha, \beta \in S$ and $\alpha \neq \beta$.

Let us construct a subalgebra of $\mathfrak{g}$. Let $\alpha \in S$. Define h_α by demanding that $B(h_\alpha, H) = \alpha(H)$ for every $H \in \mathfrak{a}$. Put $H_\alpha = 2h_\alpha / B(h_\alpha, h_\alpha)$ then we have $\alpha(H_\alpha) = 2$. Next, let us fix an element $X_\alpha \in \mathfrak{g}_\alpha$ with $[X_\alpha, \theta X_\alpha] = -H_\alpha$. Because $\dim(\mathfrak{g}_\alpha) = 1$ there are just two possible choices which differ by a sign. Put $X_{-\alpha} = -\theta X_\alpha$ then $H_\alpha, X_\alpha, X_{-\alpha}$ is a standard $\mathrm{sl}(2, \mathbb{R})$ triple. They generate a Lie subalgebra $\mathfrak{g}(\alpha)$ of $\mathfrak{g}$. The assignments

$$(5) \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto H_\alpha, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto X_\alpha, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto X_{-\alpha}$$

define an isomorphism $\mathrm{sl}(2, \mathbb{R}) \to \mathfrak{g}(\alpha)$. The following is a special case of Helgason's 'rank-one reduction' [5] (Chapter IX, §2, p. 407). The restriction of θ to $\mathfrak{g}(\alpha)$ is a Cartan involution of $\mathfrak{g}(\alpha)$. The Cartan decomposition $\mathfrak{g}(\alpha) = \mathfrak{k}(\alpha) + \mathfrak{s}(\alpha)$ we have $\mathfrak{k}(\alpha) = \mathfrak{k} \cap \mathfrak{g}(\alpha)$ and $\mathfrak{s}(\alpha) = \mathfrak{s} \cap \mathfrak{g}(\alpha)$. The subspace $\mathfrak{a}(\alpha) := \mathbb{R}H_\alpha$ is maximal abelian in $\mathfrak{s}(\alpha)$. The corresponding root space decomposition is $\mathfrak{g}(\alpha) = \mathfrak{a}(\alpha) + \mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}$. Put $\mathfrak{n}(\alpha) = \mathfrak{g}_\alpha$ and $\bar{\mathfrak{n}}(\alpha) = \mathfrak{g}_{-\alpha}$. Define $Z_\alpha = X_\alpha + \theta X_\alpha$. Then we have $\mathfrak{k}(\alpha) = \mathbb{R}Z_\alpha$. Finally put $\mathfrak{p}(\alpha) = \mathfrak{a}(\alpha) + \mathfrak{n}(\alpha)$ then $\mathfrak{p}(\alpha)$ is a minimal parabolic subalgebra of $\mathfrak{g}(\alpha)$.

Let $G(\alpha)$, $K(\alpha)$, $A(\alpha)$ and $N(\alpha)$ be the analytic subgroups of $G(\alpha)$ with Lie algebras $\mathfrak{g}(\alpha)$, $\mathfrak{k}(\alpha)$, $\mathfrak{a}(\alpha)$ and $\mathfrak{n}(\alpha)$. Define $M(\alpha)$ and $M^*(\alpha)$ as the centralizer and the normalizer of $\mathfrak{a}(\alpha)$ in $K(\alpha)$. Let $P(\alpha)$ be the normalizer in $G(\alpha)$ of $\mathfrak{p}(\alpha)$. Helgason proved that $K(\alpha)$, $A(\alpha)$, $N(\alpha)$, $M(\alpha)$, $M^*(\alpha)$ and $P(\alpha)$ are equal to the intersection of $G(\alpha)$ with K, A, N, M, M^* and P respectively. Moreover $P(\alpha)$ is a minimal parabolic subgroup of $G(\alpha)$.

Now that we have constructed the subgroups $G(\alpha)$ we can make the connection with the theory of CW-complexes by defining the characteristic maps of the 1- and 2-cells. Let $\alpha, \beta \in S$ be different simple roots. Define the following maps:

$$(6) \qquad T_\alpha : [0,1] \to G/P, \qquad s \mapsto \exp(\pi s Z_\alpha) \cdot P$$

$$(7) \qquad T_{\alpha\beta} : [0,1]^2 \to G/P, \quad (s,t) \mapsto \exp(\pi s Z_\alpha) \exp(\pi t Z_\beta) \cdot P.$$

The central result in this section is the following theorem:

Theorem 4.1. *T_α is a characteristic map of $C(s_\alpha)$ and $T_{\alpha\beta}$ is a characteristic map of $C(s_\alpha s_\beta)$.*

The proof will be given in Section 4.2. The essential computation to obtain the boundary word of $T_{\alpha\beta}$ is the following lemma.

Lemma 4.2. $e^{\mathrm{ad}(\pi Z_\alpha)} Z_\beta = (-1)^{(\beta,\alpha^\vee)} Z_\beta$.

Proof. Let $\mathfrak{g}_{\mathbb{C}}$ be the complexification of $\mathfrak{g}$. Analogous to $\mathfrak{g}_\alpha$ and $\mathfrak{g}(\alpha)$ we construct $\mathfrak{g}_{\mathbb{C},\alpha}$ and $\mathfrak{g}_{\mathbb{C}}(\alpha)$. The isomorphism $\mathrm{sl}(2,\mathbb{R}) \to \mathfrak{g}(\alpha)$ defined by (5) extends to an isomorphism $\phi : \mathrm{sl}(2,\mathbb{C}) \to \mathfrak{g}_{\mathbb{C}}(\alpha)$. Observe that $\mathrm{ad}\circ\phi$ is a representation of $\mathrm{sl}(2,\mathbb{C})$ in $\mathfrak{g}_{\mathbb{C}}$. Because $\mathrm{SL}(2,\mathbb{C})$ is simply connected, this representation lifts to a representation Ψ of $\mathrm{SL}(2,\mathbb{C})$ in $\mathfrak{g}_{\mathbb{C}}$. In particular we have $e^{\mathrm{ad}\,\phi(X)} = \Psi(\exp(X))$ for any $X \in \mathrm{sl}(2,\mathbb{C})$. For $X \in \mathfrak{g}_{\mathbb{C}}(\alpha)$ write $X' = \phi^{-1}(X) \in \mathrm{sl}(2,\mathbb{C})$ then we see that

$$e^{\mathrm{ad}\,(\pi Z_\alpha)} = e^{\mathrm{ad}\,(\phi(\pi Z'_\alpha))} = \Psi(\exp(\pi Z'_\alpha)).$$

Now we observe that in $\mathrm{SL}(2,\mathbb{C})$

$$\begin{aligned}\exp(\pi Z'_\alpha) &= \exp\begin{pmatrix} 0 & \pi \\ -\pi & 0\end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1\end{pmatrix}\\ &= \exp\begin{pmatrix} \pi i & 0 \\ 0 & -\pi i\end{pmatrix} = \exp(\pi i H'_\alpha).\end{aligned}$$

Thus we have

$$\begin{aligned} e^{\mathrm{ad}(\pi Z_\alpha)} &= \Psi(\exp(\pi i H'_\alpha))\\ &= e^{\mathrm{ad}(\phi(\pi i H'_\alpha))}\\ &= e^{\mathrm{ad}(\pi i H_\alpha)}.\end{aligned}$$

Now we claim that the restriction of $e^{\mathrm{ad}(\pi i H_\alpha)}$ to $\mathfrak{g}_{\mathbb{C},\pm\beta}$ is $(-1)^{(\beta,\alpha^\vee)}$ times the identity. Indeed for $X \in \mathfrak{g}_{\mathbb{C},\pm\beta}$ we have:

$$e^{\mathrm{ad}(\pi i H_\alpha)} X = e^{\pm \pi i(\beta,\alpha^\vee)} X = (-1)^{(\beta,\alpha^\vee)} X.$$

This proves the lemma. $\square$

Write t_α for the homotopy class of T_α in the 1-skeleton and let $\varepsilon = \varepsilon(\alpha,\beta)$ be the integer $(-1)^{(\beta,\alpha^\vee)}$ which is either $+1$ or -1 depending on the parity of $(\beta,\alpha^\vee)$. Note that the matrix $((\beta,\alpha^\vee))_{\alpha,\beta}$ is the Cartan matrix of the root system Σ.

Corollary 4.3. *A boundary word of $T_{\alpha\beta}$ is $t_\beta t_\alpha t_\beta^{-\varepsilon} t_\alpha^{-1}$.*

Proof. Define the paths $\phi_i : [0,1] \to \partial([0,1]^2)$ for $i = 1 \ldots 4$ as follows: $\phi_1(s) = (0,s)$, $\phi_2(s) = (s,1)$, $\phi_3(s) = (1,1-s)$ and $\phi_4(s) = (1-s,0)$. For an illustration see figure 1.

The concatenation of paths $\phi = \phi_1 \star \phi_2 \star \phi_3 \star \phi_4$ is a loop in $\partial([0,1]^2)$ generating its fundamental group. For $i = 1, \ldots, 4$ the paths $T_{\alpha\beta} \circ \phi_i : [0,1] \to G/P$ are loops in the 1-skeleton of G/P. Therefore we have $T_{\alpha\beta} \circ \phi = (T_{\alpha\beta} \circ \phi_1) \star (T_{\alpha\beta} \circ \phi_2) \star (T_{\alpha\beta} \circ \phi_3) \star (T_{\alpha\beta} \circ \phi_4)$. By using the definition of T_α and T_β we see that

- $T_{\alpha\beta} \circ \phi_1 = T_\beta$,
- $T_{\alpha\beta} \circ \phi_2 = T_\alpha$,
- $T_{\alpha\beta} \circ \phi_4 = T_\alpha^{-1}$.

For the loop $T_{\alpha\beta} \circ \phi_3$ we compute:

$$\begin{aligned}
(T_{\alpha\beta} \circ \phi_3)(s) &= \exp(\pi Z_\alpha) \exp(-\pi s Z_\beta) \cdot P \\
&= \exp(-\pi s \operatorname{Ad}(\exp(\pi Z_\alpha)) Z_\beta) \cdot P \\
&= \exp(-\pi s e^{\operatorname{ad} \pi Z_\alpha} Z_\beta) \cdot P \qquad (8) \\
&= \exp(-\varepsilon(\alpha, \beta) \pi s Z_\beta) \cdot P \qquad (9) \\
&= T_\beta^{-\varepsilon(\alpha,\beta)}(s).
\end{aligned}$$

To obtain (9) from (8) we used Lemma 4.2.

Now we have for the boundary loop of $T_{\alpha\beta} : T_{\alpha\beta} \circ \phi = T_\beta \star T_\alpha \star T_\beta^{-\varepsilon} \star T_\alpha^{-1}$. Therefore the boundary word is $t_\beta t_\alpha t_\beta^{-\varepsilon} t_\alpha^{-1}$.

4.2. Proof of Theorem 4.1

First, let us describe the 1- and 2-cells more conveniently.

Lemma 4.4. *Let $\alpha, \beta \in S$ with $\alpha \neq \beta$, then*

(i) $C(s_\alpha) = N(\alpha) s_\alpha \cdot P$.
(ii) $C(s_\alpha s_\beta) = N(\alpha) N(s_\alpha(\beta)) s_\alpha s_\beta \cdot P$.
(iii) $k_\alpha \in K(\alpha) \backslash M(\alpha) \Leftrightarrow k_\alpha \cdot P \in C(s_\alpha)$.

Proof. Part (i) and (ii) are a direct consequence of Lemma 2.3 in [3]. For part (iii) we used the isomorphism $K(\alpha)/M(\alpha) \simeq G(\alpha)/P(\alpha)$. □

We know that $M(\alpha) = \{e, \exp(\pi Z_\alpha)\}$, therefore Lemma 4.4 implies the following:

(i) $T_\alpha(0) = T_\alpha(1) = e \cdot P$ are in the 0-skeleton.
(ii) T_α is injective on $]0, 1[$ and maps it onto $C(s_\alpha)$.

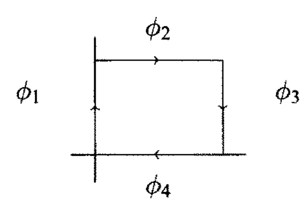

Fig. 1. Illustration of ϕ.

Here $]0,1[$ is the interior of $[0,1]$. It can be shown that the restriction of T_α to $]0,1[$ is a diffeomorphism. Now we can conclude that T_α is a characteristic map of $C(s_\alpha)$.

To prepare the proof the second statement of Theorem 4.1 we need a technical lemma.

Lemma 4.5. *Let α and β be simple roots with $\alpha \neq \beta$. Then:*

(i) *MA normalizes $N(\alpha)$.*

(ii) *M normalizes $K(\alpha)$ hence $M(\alpha)$,*

(iii) *for any $n_\alpha \in N(\alpha)$ and $n_\beta \in N(\beta)$ we have $n_\alpha n_\beta s_\beta \cdot P = n_\beta s_\beta \cdot P$.*

Proof. (i) and (ii) are trivial. For (iii) write $n_\alpha = \exp(X_\alpha)$ and $n_\beta = \exp(X_\beta)$. We have

$$\begin{aligned} n_\beta^{-1} n_\alpha n_\beta &= \exp(\mathrm{Ad}(n_\beta^{-1})X_\alpha) \\ &= \exp(e^{\mathrm{ad}(-X_\beta)} X_\alpha). \end{aligned}$$

Observe that $e^{\mathrm{ad}(-X_\beta)} X_\alpha \in \sum_{j \geq 0} \mathfrak{g}_{\alpha + j\beta}$. Because $s_\beta(\alpha + j\beta)$ is a positive root for any j, we conclude that

$$s_\beta^{-1} n_\beta^{-1} n_\alpha n_\beta s_\beta \in N \subset P,$$

which proves part (iii). $\square$

Proof (of Theorem 4.1). Only the last part is left to prove.

Consider the boundary of $[0,1]^2$. From the definition of $T_{\alpha\beta}$ we see that for $s, t \in [0,1]$:

(i) $T_{\alpha\beta}(s,0) = \exp(\pi s Z_\alpha) \cdot P \in V(s_\alpha)$,

(ii) $T_{\alpha\beta}(1,t) = \exp(\pi Z_\alpha)\exp(\pi t Z_\beta) \cdot P \in V(s_\beta)$ because of Lemma 4.5,

(iii) $T_{\alpha\beta}(s,1) = \exp(\pi s Z_\alpha) \cdot P \in V(s_\alpha)$,

(iv) $T_{\alpha\beta}(0,t) = \exp(\pi t Z_\beta) \cdot P \in V(s_\beta)$.

Thus we conclude that the boundary of $[0,1]^2$ is mapped into the 1-skeleton.

Next we will show that $]0,1[^2$ is mapped surjectively onto $C(s_\alpha s_\beta)$. Let (s,t) be in $]0,1[^2$, then according to Lemma 4.5 we have $\exp(\pi s Z_\alpha) = n_\alpha s_\alpha q \tilde{n}_\alpha$ and $\exp(\pi t Z_\beta) \cdot P = n_\beta s_\beta \cdot P$ for certain $q \in MA$, $n_\beta \in N(\beta)$ and $n_\alpha, \tilde{n}_\alpha \in N(\alpha)$. When we apply this to $T_{\alpha\beta}$ we obtain the following:

$$\begin{aligned} T_{\alpha\beta}(s,t) &= n_\alpha s_\alpha q \tilde{n}_\alpha n_\beta s_\beta \cdot P, \text{ using Lemma 4.5 (iii):} \\ &= n_\alpha s_\alpha q n_\beta s_\beta \cdot P, \text{ using Lemma 4.5 (ii) with } n'_\beta = q n_\beta q^{-1} \in N(\beta): \\ &= n_\alpha s_\alpha n'_\beta q s_\beta \cdot P, \text{ using that } M^* \text{ normalizes } MA: \\ &= n_\alpha s_\alpha n'_\beta s_\beta \cdot P, \text{ finally, taking } n_{s_\alpha(\beta)} = s_\alpha n'_\beta s_\alpha^{-1} \in N(s_\alpha(\beta)): \\ &= n_\alpha n_{s_\alpha(\beta)} s_\alpha s_\beta \cdot P. \end{aligned}$$

From Lemma 4.4 it is clear that this is an element of the 2-cell $C(s_\alpha s_\beta)$. Reversing this argument shows that every element of $C(s_\alpha s_\beta)$ is reached.

Now we shall prove that $T_{\alpha\beta}$ is injective on $]0,1[^2$. Let (s,t), $(\tilde{s},\tilde{t}) \in]0,1[^2$

and assume $T_{\alpha\beta}(s,t) = T_{\alpha\beta}(\tilde{s},\tilde{t})$. Multiplying from the left with $\exp(-\pi\tilde{s}Z_\alpha)$ we get

$$(10) \qquad T_{\alpha\beta}(s-\tilde{s},t) = T_{\alpha\beta}(0,\tilde{t}).$$

The right hand side of Equation (10) is an element of $V(s_\beta)$. This implies that $s=\tilde{s}$, otherwise the left hand side is an element of $C(s_\alpha s_\beta)$ which is disjoint from $V(s_\beta)$. It follows that $t=\tilde{t}$ and this proves that $T_{\alpha\beta}$ is injective on $]0,1[^2$. It can also be proved that the restriction of $T_{\alpha\beta}$ to $]0,1[^2$ is a diffeomorphism. With this we can conclude that $T_{\alpha\beta}$ is a characteristic map of the 2-cell $C(s_\alpha s_\beta)$ and we have proved Theorem 4.1. □

Remark 4.6. The situation is even more beautiful! Let α and β different simple roots with multiplicity 1. The closure of a 2-cell $V(s_\alpha s_\beta)$ is a smooth manifold.

The group $M(\alpha)$ acts on the smooth manifold $K(\alpha) \times (K(\beta)/M(\beta))$ in the following way:

$$m_\alpha \cdot (k_\alpha, \overline{k_\beta}) := (k_\alpha m_\alpha^{-1}, \overline{m_\alpha k_\beta m_\alpha^{-1}}).$$

This is a free smooth action and therefore the orbit space $\Gamma_{\alpha\beta}$ is a smooth manifold. The product map $K(\alpha)\times K(\beta) \to G/P$ induces an immersion $\psi : \Gamma_{\alpha\beta} \to G/P$. $\Gamma_{\alpha\beta}$ as well as its image $V(s_\alpha s_\beta)$ is compact, therefore ψ is a diffeomorphism. The projection $K(\alpha) \times (K(\beta)/M(\beta)) \to K(\alpha)$ induces a fibration $\Gamma_{\alpha\beta} \to V(s_\alpha)$ and the fiber is $V(s_\beta)$.

So, the closure of the 2-cell $\Gamma_{\alpha\beta} \simeq V(s_\alpha s_\beta)$ is a circle bundle over a circle. Up to homotopy there are only two kinds: a trivial one (2-torus) and a non-trivial one (Klein Bottle). The Cartan integer $(\beta,\alpha^\vee)$ determines which one it is: the bundle is trivial exactly when $(\beta,\alpha^\vee)$ is even and it is non-trivial exactly when $(\beta,\alpha^\vee)$ is odd.

4.3. Proof of Theorem 1.1

Theorem 4.1 holds for arbitrary G and P and immediately implies Theorem 1.1 if G is split and P is minimal. We will now show that Theorem 4.1 implies Theorem 1.1 in the general case.

When G is not necessarily split and F is not necessarily empty, then there are possibly even less 1-cells: only $C(s_\alpha)$ with $\alpha \in S^*\backslash F$. In this case there are three types of 2-cells:

(i) $C(s_\alpha)$ with $\alpha \in S\backslash F$, $m_\alpha = 2$ and $m_{2\alpha} = 0$. Its boundary in G/P_F is the 0-cell $C(e) = e\cdot P$ therefore its boundary loop (hence its boundary word) is trivial. (The closure of this 2-cell, $V(s_\alpha)$, is a 2-sphere.)

(ii) $C(s_\alpha s_\beta)$ with $\alpha,\beta \in S^*\backslash F$ and $\alpha \neq \beta$. This is the case of Theorem 4.1 and we see that $t_\beta t_\alpha t_\beta^{-\varepsilon(\alpha,\beta)} t_\alpha^{-1}$ is a boundary word. (The closure of this 2-cell is a 2-torus or a Klein Bottle.)

(iii) $C(s_\alpha s_\beta)$ with $\alpha \in S^* \cap F$, $\beta \in S^* \backslash F$ and $\alpha \not\perp \beta$. In this case we can compute the boundary loop in the same way as in Corollary 4.3. However, as $\alpha \in F$, the loop T_α is trivial and we see that $t_\beta t_\beta^{-\varepsilon(\alpha,\beta)}$ is a boundary word. After adding a generator t_α (which does not correspond to a 1-cell) this can also be represented as the relations $t_\beta t_\alpha t_\beta^{-\varepsilon(\alpha,\beta)} t_\alpha^{-1} = e$ and $t_\alpha = e$. This may seem rather inefficient but it will make the presentation more elegant. (The closure of this 2-cell is a pinched 2-torus or a pinched Klein Bottle.)

The previous remarks show that the presentation given in the introduction is a presentation of the fundamental group of G/P_F. This proves Theorem 1.1. □

5. APPLICATION

In contrast with an arbitrary presentation, the current presentation is easy to handle.

Assume we are in the situation of Section 4.1. When $\alpha \perp \beta$ then t_α and t_β commute, therefore we may restrict ourselves to connected Dynkin diagrams. The following two rank-2 situations are the essential ones:

5.1. A_2 and G_2

Let α and β be the simple roots. This bond between α and β in the Dynkin diagram of $\mathfrak{g}$, is responsible for all the non-commutativity in the fundamental group. The generators are t_α and t_β. The relations are:

$$\begin{cases} t_\alpha t_\beta = t_\beta t_\alpha^{-1} \\ t_\beta t_\alpha = t_\alpha t_\beta^{-1}. \end{cases}$$

Call the resulting group H. From this presentation one deduces that for the commutator c of t_α and t_β we have:

$$c = t_\alpha^2 = t_\beta^2 = t_\alpha^{-2} = t_\beta^{-2}, \quad c^2 = e.$$

So the center $C = \{e, c\}$ is a cyclic group of order 2. Clearly the group H/C consists of 4 elements and is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Consequently H has 8 elements and is the so called quaternion group, which was already observed by Ehresmann [4].

5.2. B_2

Again, let α and β be the simple roots, and assume β is the shortest root. The generators are t_α and t_β and the relations are:

$$\begin{cases} t_\alpha t_\beta = t_\beta t_\alpha \\ t_\beta^2 = e. \end{cases}$$

Clearly the resulting group is isomorphic to $\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$.

Table 1. $\pi_1(G/P) = \pi_1$ when P is minimal and G is split and simple

Type		π_1	$\lvert\pi_1\rvert$
A_n	$(n \geq 2)$	$Q(n)$	2^{n+1}
B_n	$(n \geq 3)$	$Q(n-1) \times (\mathbb{Z}/2\mathbb{Z})$	2^{n+1}
C_n	$(n \geq 1)$	$(\mathbb{Z}/2\mathbb{Z})^{n-1} \times \mathbb{Z}$	∞
F_4		$Q(2) \times (\mathbb{Z}/2\mathbb{Z})^2$	32

5.3. General case

With these calculations we can determine the fundamental group of G/P. In Table 1 the fundamental groups are listed when G is split, simple and of type A, B, C or F. The group $Q(n)$ is defined by n generators $t_1, \dots, t_n$ with relations:

$$\begin{cases} t_i t_j = t_j t_i & \text{for } |i-j| > 1 \\ t_i t_j = t_j t_i^{-1} & \text{for } |i-j| = 1. \end{cases}$$

Then we have $Q(1) \simeq \mathbb{Z}$ and $Q(2)$ is the quaternion group.
In this case M is a discrete group which has the presentation:

$$\langle m_\alpha, \alpha \in S \,|\, m_\alpha^2 = e, \quad m_\alpha m_\beta = m_\beta m_\alpha \,\forall\, \alpha, \beta \in S \rangle.$$

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